§3.2 Background Field Gauge

Consider the effective action

$$\Gamma[A, \psi, \omega, \omega^*]$$

as a functional of <u>classical</u> external gauge, matter, ghost and antighost fields:

$A_{\alpha\mu}(x)$, $\psi_\ell(x)$, $\omega_\alpha(x)$, $\omega^*_\alpha(x)$

→ calculated by replacing <u>quantum</u> fields

A', ψ', ω', $\omega^{*'}$ by shifted fields

$A + A'$, $\psi + \psi'$, $\omega + \omega'$, $\omega^* + \omega'^*$

→ path integral is taken over primed fields with unprimed ones held fixed

Choose gauge fixing function

$$f_\alpha = \partial_\mu A'^\mu_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} A'^\mu_\gamma ,$$

→ $f_\alpha f_\alpha$ invariant under transformations

$$\begin{aligned} \delta A^\mu_\alpha &= \partial^\mu \varepsilon_\alpha - C_{\alpha\beta\gamma} \varepsilon_\beta A^\mu_\gamma \\ \delta A'^\mu_\alpha &= - C_{\alpha\beta\gamma} \varepsilon_\beta A'^\mu_\gamma \end{aligned} \qquad (1a)$$

<u>Check:</u> $f_\alpha = \bar{D}_\alpha A'^\mu_\alpha$, with $\bar{D}_\mu = \partial_\mu + C_{\alpha\beta\gamma} A_\beta$

Thus A'^μ_α behaves as an ordinary matter field in adjoint representation and we have $\delta f_\alpha = -C_{\alpha\beta\gamma}\,\varepsilon_\beta f_\gamma$

$\longrightarrow \delta(f_\alpha f_\alpha) = -2C_{\alpha\beta\gamma} f_\alpha \varepsilon_\beta f_\gamma = 0$

<u>But</u> $f_\alpha f_\alpha$ is <u>not</u> invariant under the true gauge transformations

$$\delta_{true} A_\alpha^\mu = 0 \qquad \text{(since background field)}$$

(2a)

$$\delta_{true} A_\alpha'^\mu = \partial^\mu \varepsilon_\alpha - C_{\alpha\beta\gamma}\,\varepsilon_\beta (A_\gamma^\mu + A_\gamma'^\mu)$$
$$= \bar{D}_\mu \varepsilon_\alpha - C_{\alpha\beta\gamma}\,\varepsilon_\beta A_\gamma'^\mu$$

giving $\delta_{true} f_\alpha = \bar{D}_\mu (\bar{D}^\mu \varepsilon_\alpha - C_{\alpha\beta\gamma}\,\varepsilon_\beta A_\gamma'^\mu)$

$\longrightarrow f_\alpha f_\alpha$ serves as gauge fixing term

Transformations (1a) and (2a) are equivalent when acting on other terms in the Lagrangian!

$$\delta(A_\alpha^\mu + A_\alpha'^\mu) = \delta_{true}(A_\alpha^\mu + A_\alpha'^\mu)$$
$$= \partial^\mu \varepsilon_\alpha - C_{\alpha\beta\gamma}\,\varepsilon_\beta (A_\gamma^\mu + A_\gamma'^\mu)$$

Defining matter field trfs.

$$\delta\psi = i t_\alpha \varepsilon_\alpha \psi, \qquad\qquad (1b)$$
$$\delta\psi' = i t_\alpha \varepsilon_\alpha \psi',$$

then also $\delta(\psi + \psi') = i t_\alpha \varepsilon_\alpha (\psi + \psi')$

Note that
$$\delta_{true}\psi = 0,$$
$$\delta_{true}\psi' = it_\alpha \epsilon_\alpha(\psi+\psi') \qquad (2b)$$
Write original Lagrangian as
$$\mathcal{L} = -\frac{1}{4}\Big(\partial_\mu[A_{\alpha\nu}+A'_{\alpha\nu}] - \partial_\nu[A_{\alpha\mu}+A'_{\alpha\mu}]$$
$$+ C_{\alpha\beta\gamma}[A_{\beta\mu}+A'_{\beta\mu}][A_{\gamma\nu}+A'_{\gamma\nu}]\Big)^2$$
$$+ \mathcal{L}_M\big(\psi+\psi', \partial_\mu(\psi+\psi') - it_\alpha(A_{\alpha\mu}+A'_{\alpha\mu})(\psi+\psi')\big)$$
$$= -\frac{1}{4}\left(F_{\alpha\mu\nu} + \bar{D}_\mu A'_{\alpha\nu} - \bar{D}_\nu A'_{\alpha\mu} + C_{\alpha\beta\gamma}A'_{\beta\mu}A'_{\gamma\nu}\right)^2$$
$$+ \mathcal{L}_M\left(\psi+\psi', \bar{D}_\mu(\psi+\psi') - it_\alpha A'_{\alpha\mu}(\psi+\psi')\right),$$

where
$$\bar{D}_\mu A'_{\alpha\nu} \equiv \partial_\mu A'_{\alpha\nu} + C_{\alpha\beta\gamma}A_{\beta\mu}A'_{\gamma\nu},$$
$$\bar{D}_\mu \psi \equiv \partial_\mu\psi - it_\alpha A_{\alpha\mu}\psi,$$
and $F_{\alpha\mu\nu}$ is the background field strength
$$F_{\alpha\mu\nu} \equiv \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} + C_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu}$$

$\rightarrow$ $\mathcal{L}$ is ivariant under trfs. (1a), (1b)

Let us now consider $\mathcal{L}_{GH} = \omega^*_\alpha \Delta_\alpha$,

where $\Delta_\alpha = \bar{D}_\mu\left[\bar{D}_\mu(\omega_\alpha+\omega'_\alpha) - C_{\alpha\beta\gamma}(\omega_\beta+\omega'_\beta)A'^{\mu}_\gamma\right]$

or integrating by parts,

$\mathcal{L}_{GH}$

$$= -\left(\bar{D}_\mu(\omega_\alpha^* + \omega_\alpha'^*)\right]\left(\bar{D}^\mu(\omega_\alpha + \omega_\alpha') - C_{\alpha\beta\gamma}(\omega_\beta + \omega_\beta')A_\gamma'^\mu\right)$$

This is manifestly invariant under trfs. (1a) supplemented with "matter-field-like" trfs. for ω and ω':

$$\begin{aligned} \delta\omega_\alpha &= -C_{\alpha\beta\gamma}\,\varepsilon_\beta\,\omega_\gamma\,, \\ \delta\omega_\alpha' &= -C_{\alpha\beta\gamma}\,\varepsilon_\beta\,\omega_\gamma'\,, \end{aligned} \qquad (1c)$$

together with

$$\begin{aligned} \delta\omega_\alpha^* &= -C_{\alpha\beta\gamma}\,\varepsilon_\beta\,\omega_\gamma^*\,, \\ \delta\omega_\alpha'^* &= -C_{\alpha\beta\gamma}\,\varepsilon_\beta\,\omega_\gamma'^* \end{aligned} \qquad (1d)$$

Note that the combined formal trfs. (1a), (1b), (1c), and (1d) leave the complete modified Lagrangian invariant

$$\mathcal{L}_{MOD} = \mathcal{L} - \frac{1}{2\xi} f_\alpha f_\alpha + \mathcal{L}_{GH}$$

Performing the path integral over A', ψ', ω', and ω'^*, we notice that the measure

$\prod [dA'][d\psi'][d\omega'][d\omega'^*]$

is invariant under the linear trfs. (1a) to (1d) acting on primed fields.

$\rightarrow$ effective action $\Gamma[A, \psi, \omega, \omega^*]$ is invariant under (1a) to (1d) acting on unprimed fields $A, \psi, \omega, \omega^*$!

$\rightarrow$ can fix the terms of Γ using this gauge invariance:

Note that coefficients of terms in Γ have dimensionalities $[\text{mass}]^d$, $d \leq 4$ by power-counting renormalizability.

$\rightarrow$ invariance under (1a) – (1d) gives:

$$\Gamma_\infty = \int d^4x\, \mathcal{L}_\infty ,$$

$$\mathcal{L}_\infty = -\frac{1}{4} L_A F_{\alpha\mu\nu} F_\alpha{}^{\mu\nu} - L_\psi \bar{\psi} \gamma^\mu \bar{D}_\mu \psi - m L_m \bar{\psi}\psi - L_\omega (\bar{D}_\mu \omega_\alpha^*)(\bar{D}^\mu \omega_\alpha),$$

where $F_{\alpha\mu\nu}$, $\bar{D}_\mu \psi$, $\bar{D}_\mu \omega_\alpha$, and $\bar{D}_\mu \omega_\alpha^*$ are given in terms of background fields:

$$F_{\alpha\mu\nu} \equiv \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} + C_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu},$$

$$\bar{D}_\mu \psi \equiv \partial_\mu \psi - i t_\alpha A_{\alpha\mu} \psi,$$

$$\bar{D}_\mu \omega_\alpha \equiv \partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma,$$

$$\bar{D}_\mu \omega_\alpha^* \equiv \partial_\mu \omega_\alpha^* + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma^*$$

→ dimensional analysis tells us that

$$[L_A] = [L_\psi] = [L_m] = [L_\omega] = \Lambda^0$$

→ expect them to be logarithmically divergent

Defining the classical Lagrangian

$$\mathcal{L}_{CLASS} = -\frac{1}{4} F_{\alpha\mu\nu} F_\alpha^{\mu\nu} - \overline{\psi}(\gamma^\mu \mathcal{D}_\mu + m)\psi - (\mathcal{D}_\mu \omega_\alpha^*)(\mathcal{D}^\mu \omega_\alpha),$$

we obtain

$$\mathcal{L}_{CLASS} + \mathcal{L}_{co} = -\frac{1}{4} F^R_{\alpha\mu\nu} F_\alpha^{R\mu\nu} - \overline{\psi}^R \gamma^\mu D^R_\mu \psi^R - m^R \overline{\psi}^R \psi^R - (D^R_\mu \omega_\alpha^{*R})(D^{\mu R} \omega_\alpha^R),$$

where

$$A^R_{\alpha\mu} \equiv \sqrt{1+L_A}\, A_{\alpha\mu},$$

$$\psi^R_\ell \equiv \sqrt{1+L_\psi}\, \psi_\ell,$$

$$\omega^R_\alpha \equiv \sqrt{1+L_\omega}\, \omega_\alpha,$$

$$\omega^{R*}_\alpha \equiv \sqrt{1+L_\omega}\, \omega^*_\alpha,$$

$$m^R \equiv m(1+L_m)/(1+L_\psi)$$

and renormalized structure constants, group generators $C^R_{\alpha\beta\gamma} \equiv (1+L_A)^{-1/2} C_{\alpha\beta\gamma}$,

$$t_\alpha^R \equiv (1+L_A)^{-1/2} t_\alpha$$

From the above we deduce renormalized gauge coupling constant:

$$g^R = g(1 + L_A)^{-1/2}$$